Paraproducts : Sparse Domination 2 Ways

~ Prelude ~

→ (main) Paraproducts on $\mathbb{R}^n$: $\Pi_b f = \sum_{Q\in\mathcal{D}} (b, h_Q)\langle f\rangle_Q h_Q$

$\Pi_b^* f = \sum_{Q\in\mathcal{D}} (b, h_Q)(f, h_Q)\frac{1\!\!1_Q}{|Q|}$

→ **The L^2-bound for Π_b (unweighted case):**

The simplest proof appeals to $\mathcal{H}^1_\mathcal{D}$ – $BMO_\mathcal{D}$ duality (will prove this later):

$$|(b,\phi)| \lesssim \|b\|_{BMO_\mathcal{D}} \|\phi\|_{\mathcal{H}^1_\mathcal{D}} \quad \text{where} \quad \boxed{\|\phi\|_{\mathcal{H}^1_\mathcal{D}} := \|S_\mathcal{D}\phi\|_{L^1}}$$

So take $f, g \in L^2(dx)$:

$$|(\Pi_b f, g)| = \Big|\sum_Q (b,h_Q)\langle f\rangle_Q (g,h_Q)\Big| = \Big|\Big(b, \underbrace{\sum_Q \langle f\rangle_Q (g,h_Q)h_Q}_{=:\phi}\Big)\Big|$$

$$\lesssim \|b\|_{BMO_\mathcal{D}} \|S_\mathcal{D}\phi\|_{L^1}$$

now:

$$S^2_\mathcal{D}\phi = \sum_Q \langle f\rangle_Q^2 (g,h_Q)^2 \frac{1\!\!1_Q}{|Q|} \leq (M_\mathcal{D} f)^2 (S_\mathcal{D} g)^2 \Rightarrow \|S_\mathcal{D}\phi\|_{L^1} \leq \int (M_\mathcal{D} f)(S_\mathcal{D} g)$$

$$\leq \|M_\mathcal{D} f\|_{L^2} \|S_\mathcal{D} g\|_{L^2}$$

$$\lesssim \|f\|_{L^2}\|g\|_{L^2}$$

$$\Rightarrow |(\Pi_b f, g)| \lesssim \|b\|_{BMO_\mathcal{D}} \|f\|_{L^2}\|g\|_{L^2}$$

$$\Rightarrow \|\Pi_b : L^2 \to L^2\| \lesssim \|b\|_{BMO_\mathcal{D}} \quad \text{(these are actually equivalent)}$$

→ **Weighted case?** Nothing stops this argument from going through for $\Pi_b : L^2(w) \to L^2(w)$ where $w \in A_2$: Take $f \in L^2(w)$, $g \in L^2(w^{-1})$:

$$|(\Pi_b f, g)| \lesssim \|b\|_{BMO_\mathcal{D}} \underbrace{\|S_\mathcal{D}\phi\|_{L^1}}$$

$$\leq \int (M_\mathcal{D} f)(S_\mathcal{D} g) = \int (M_\mathcal{D} f) w^{1/2} (S_\mathcal{D} g) w^{-1/2}$$

$$\leq \|M_\mathcal{D} f\|_{L^2(w)} \|S_\mathcal{D} g\|_{L^2(w^{-1})}$$

$$\lesssim [w]^2_{A_2} \|f\|_{L^2(w)} \|g\|_{L^2(w^{-1})}$$

So we obtain that Π_b is bounded $L^2(w) \to L^2(w)$, but appealing to $M_\mathcal{D}$ and $S_\mathcal{D}$ each produces a $[w]_{A_2}$. The sharp bound is actually <u>linear</u> in A_2.

Side note (**Homework**): Show that $\|b\|_{BMO_\mathcal{D}} \lesssim \|\Pi_b : L^2 \to L^2\|$.

<u>Hint</u>: Show that

$$(b - \langle b\rangle_Q) 1\!\!1_Q = (\Pi_b 1\!\!1_Q - \Pi_b^* 1\!\!1_Q) 1\!\!1_Q.$$

(Pointwise) Sparse Domination for Paraproducts

→ Inspired by Lacey's proof of pointwise domination for the martingale transform (M.T. Lacey – "An elementary proof of the A_2 bound" Israel J. of Math. 2017).

→ Working in $\mathbb{R}^n$, equipped w/ a dyadic grid $\mathcal{D}$, and the paraproduct:

$$\Pi_b f(x) := \sum_{Q\in\mathcal{D}} (b, h_Q)\langle f\rangle_Q h_Q(x) \quad ; \quad b\in BMO_{\mathcal{D}}(\mathbb{R}^n).$$

As is standard in these pointwise domination proofs, we need the maximal truncation of the operator in question:

$$\boxed{\Pi_b^{\#} f(x) := \sup_{P\in\mathcal{D}} \left|\sum_{Q\supsetneq P} (b, h_Q)\langle f\rangle_Q h_Q(x)\right|}$$

(we'll need a weak (1,1) inequality for this).

→ Fix $Q_0\in\mathcal{D}$ and consider the restricted paraproduct $\boxed{\Pi_{b,Q_0} f := \sum_{Q\subset Q_0} (b, h_Q)\langle f\rangle_Q h_Q}$

Assume the following: (proof later)

(*) Let $\varepsilon\in(0,1)$. There is a constant C_0 (depending on n, ε) such that the set:

$$\boxed{F := \{x\in Q_0 : \Pi_b^{\#} f(x) > C_0\|b\|_{BMO}\langle|f|\rangle_{Q_0}\}\cup\{x\in Q_0 : M^{\mathcal{D}}_{Q_0} f(x) > C_0\langle|f|\rangle_{Q_0}\}}$$

satisfies $\boxed{|F| < \varepsilon|Q_0|}$ for all $Q_0\in\mathcal{D}$, $f\in L^1_{loc}$.

→ Let $\mathcal{F} := \{$maximal subcubes of Q_0 that are contained in $F\}$ $\Rightarrow$ $\boxed{\sum_{R\in\mathcal{F}} |R| < \varepsilon|Q_0|}$

→ We show that:

$$\boxed{\mathbb{1}_{Q_0}(x)\,|\Pi_{b,Q_0} f(x)| \lesssim 2C_0\|b\|_{BMO}\langle|f|\rangle_{Q_0}\mathbb{1}_{Q_0}(x) + \sum_{R\in\mathcal{F}} |\Pi_{b,R} f(x)|}$$

⇒ Start building a sparse collection $\mathcal{S}(Q_0)\subset\mathcal{D}(Q_0)$ by first adding Q_0, then $R\in\mathcal{F}$ are the $\mathcal{S}$-children of Q_0.

⇒ Recurse on the terms of the second sum (each $R\in\mathcal{F}$ plays the part of Q_0):

$\forall R\in\mathcal{F}$, build collection $\mathcal{F}(R)$ as above: $|\Pi_{b,R} f(x)| \lesssim 2C_0\|b\|_{BMO}\langle|f|\rangle_R\mathbb{1}_R(x) + \sum_{R'\in\mathcal{F}(R)} |\Pi_{b,R'} f|$

⇒ Expand the relationship above to:

$$\mathbb{1}_{Q_0}(x)|\Pi_{b,Q_0} f(x)| \lesssim 2C_0\|b\|_{BMO}\langle|f|\rangle_{Q_0}\mathbb{1}_{Q_0}(x) + 2C_0\|b\|_{BMO}\sum_{R\in ch_{\mathcal{S}}Q_0}\langle|f|\rangle_R\mathbb{1}_R(x) + \sum_{R\in ch_{\mathcal{S}}Q_0}\sum_{R'\in ch_{\mathcal{S}}R}|\Pi_{b,R'} f(x)|$$

⇒ Recursively, we really obtain:

$$\mathbb{1}_{Q_0}(x)|\Pi_{b,Q_0} f(x)| \lesssim 2C_0\|b\|_{BMO}\underbrace{\sum_{Q\in\mathcal{S}(Q_0)}\langle|f|\rangle_Q\mathbb{1}_Q(x)}_{=\mathcal{A}_{\mathcal{S}(Q_0)}|f|(x)} \;!$$

→ 2 remarks about the collection $\mathcal{S}(Q_0)$:

→ $\mathcal{S}(Q_0)$ satisfies the $\mathcal{S}$-children definition of sparseness, with $\varepsilon\in(0,1)$ ⇒ $\mathcal{S}(Q_0)$ is $(1-\varepsilon)$-sparse!

→ $\mathcal{S}(Q_0)$ depends on f and b (the specific cubes that go into $\mathcal{F}$ depend on b & f but the constant C_0 is universal) but that is OK!

Theorem:

Let $\eta \in (0,1)$. There is a constant C (depending on η and the dimension n) such that:

For every $b \in BMO_{\mathcal{D}}(\mathbb{R}^n)$, $f \in L^1_{loc}$ and $Q_0 \in \mathcal{D}$, there is an η-sparse collection $\mathcal{S}(Q_0) \subset \mathcal{D}(Q_0)$ such that:

$$\boxed{|\Pi_{b,Q_0} f(x)| \leq C \|b\|_{BMO}\, \mathcal{A}_{\mathcal{S}(Q_0)} |f|(x)} \quad \forall x \in Q_0$$

So suppose $b \in BMO_{\mathcal{D}}(\mathbb{R}^n)$ has <u>finite Haar expansion</u>. Then there are at most 2^n disjoint dyadic cubes $\{Q_K\}_{1 \leq K \leq 2^n}$ such that:

$$b = \sum_K \sum_{Q \subset Q_K} (b, h_Q) h_Q$$

and then

$$\Pi_b f = \sum_K \Pi_{b,Q_K} f$$

$\Rightarrow$ for each Q_K, there is an η-sparse collection $\mathcal{S}(Q_K) \subset \mathcal{D}(Q_K)$ s.t.

$$|\Pi_b f(x)| \leq \sum_K |\Pi_{b,Q_K} f(x)| \leq C \|b\|_{BMO} \sum_K \mathcal{A}_{\mathcal{S}(Q_K)} |f|(x)$$
$$= C \|b\|_{BMO} \mathcal{A}_{\mathcal{S}} |f|(x)$$

where $\mathcal{S} := \bigcup_{K=1}^{2^n} \mathcal{S}(Q_K)$ is also η-sparse.

$$\Rightarrow \|\Pi_b f\|_{L^2(w)} \leq C \|b\|_{BMO} \|\mathcal{A}_{\mathcal{S}} |f|\|_{L^2(w)}$$
$$\lesssim \|b\|_{BMO} [w]_{A_2} \|f\|_{L^2(w)}$$

$\Rightarrow \|\Pi_b : L^2(w) \to L^2(w)\| \lesssim \|b\|_{BMO} [w]_{A_2}$ holds for all $b \in BMO_{\mathcal{D}}$ w/ finite Haar expansion
$\Rightarrow$ also holds for <u>all</u> $b \in BMO_{\mathcal{D}}$!

$\Rightarrow$ <u>linear A_2 bound for paraproducts</u> (first obtained by Beznosova using a weighted CET and Bellman functions).

<u>Remark</u>: The adjoint $\Pi_b^* : L^2(w') \to L^2(w')$; $\Pi_b^* f = \sum_Q (b, h_Q)(f, h_Q) \frac{\mathbb{1}_Q}{|Q|}$ then also automatically follows the linear bound.

$F = \{x \in Q_0 : \overset{\triangleright}{\Pi}_b f(x) > C_0 \|b\|_{BMO} \langle |f| \rangle_{Q_0}\} \cup \{x \in Q_0 : M^{\mathcal{D}}_{Q_0} f(x) > C_0 \langle |f| \rangle_{Q_0}\}$

$\mathcal{F} := \{\text{maximal subcubes of } Q_0 \text{ contained in } F\}$

$\Pi_{b,Q_0} f(x) = \sum_{Q \subset Q_0} (b, h_Q) \langle f \rangle_Q h_Q(x).$

Case 1: $\boxed{x \notin F}$ $\Rightarrow \overset{\triangleright}{\Pi}_b f(x) \leq C_0 \|b\|_{BMO} \langle |f| \rangle_{Q_0}.$

$\to$ $\overset{\triangleright}{\Pi}_b$ **dominates** Π_b: $|\Pi_b f(x)| \leq \overset{\triangleright}{\Pi}_b f(x), \forall x \in \mathbb{R}^n$:

Let $x \in \mathbb{R}^n$. Then $\Pi_b f(x) = \sum_{Q \in \mathcal{D}} (b, h_Q) \langle f \rangle_Q h_Q(x) = \sum_{K \in \mathbb{Z}} (b, h_{Q_K}) \langle f \rangle_{Q_K} h_{Q_K}(x)$,

where for every $K \in \mathbb{Z}$, Q_K is the unique cube in $\mathcal{D}$ of side length 2^K that contains x.

Fix $m \in \mathbb{Z}$:

$$\left| \sum_{K > m} (b, h_{Q_K}) \langle f \rangle_{Q_K} h_{Q_K}(x) \right| = \left| \sum_{Q \supsetneq Q_m} (b, h_Q) \langle f \rangle_Q h_Q(x) \right| \leq \overset{\triangleright}{\Pi}_b f(x)$$

Taking $m \to -\infty$: $|\Pi_b f(x)| \leq \overset{\triangleright}{\Pi}_b f(x)$. ☒

$\to$ For $x \in Q_0$: $\Pi_b f(x) = \sum_{Q \in \mathcal{D}} (b, h_Q) \langle f \rangle_Q h_Q(x) = \sum_{Q \subset Q_0} (b, h_Q) \langle f \rangle_Q h_Q(x) + \sum_{Q \supsetneq Q_0} (b, h_Q) \langle f \rangle_Q h_Q(x)$

Q must intersect Q_0

$\Pi_{b,Q_0} f(x)$

$= \sum_{Q \supsetneq Q_0} (b, h_Q) \langle f \rangle_Q h_Q(Q_0) = \langle \Pi_b f \rangle_{Q_0}$

$\Rightarrow \Pi_{b,Q_0} f(x) = \Pi_b f(x) - \langle \Pi_b f \rangle_{Q_0}, \forall x \in Q_0$

$\Rightarrow$ If $x \notin F$:

$$|\Pi_{b,Q_0} f(x)| \leq |\Pi_b f(x)| + |\langle \Pi_b f \rangle_{Q_0}|$$
$$\leq \overset{\triangleright}{\Pi}_b f(x) + \left| \sum_{Q \supsetneq Q_0} (b, h_Q) \langle f \rangle_Q h_Q(x) \right| \leq 2 \overset{\triangleright}{\Pi}_b f(x) \leq 2 C_0 \|b\|_{BMO} \langle |f| \rangle_{Q_0}.$$

Case 2: $\boxed{x \in F}$ $\Rightarrow$ **There is a unique $P_0 \in \mathcal{F}$ such that $x \in P_0$.**

Knowing that $x \in P_0$ means we can write $\Pi_{b,Q_0} f(x) = \sum_{\substack{Q \supsetneq P_0 \\ Q \subset Q_0}} (b, h_Q) \langle f \rangle_Q h_Q(x) + \Pi_{b,P_0} f(x)$.

Write the first term as:

$$\left| \sum_{\substack{Q \supsetneq P_0 \\ Q \subset Q_0}} (b, h_Q) \langle f \rangle_Q h_Q(x) \right| \leq \underbrace{\left| \sum_{\substack{Q \supsetneq \hat{P}_0 \\ Q \subset Q_0}} (b, h_Q) \langle f \rangle_Q h_Q(x) \right|}_{=: A(x)} + \underbrace{\left| (b, h_{\hat{P}_0}) \langle f \rangle_{\hat{P}_0} h_{\hat{P}_0}(x) \right|}_{=: B(x)}$$

$\to$ $A(y) = \left| \sum_{\substack{Q \supsetneq \hat{P}_0 \\ Q \subset Q_0}} (b, h_Q) \langle f \rangle_Q h_Q(x) \right| = \left| \sum_{\substack{Q \supsetneq \hat{P}_0 \\ Q \subset Q_0}} (b, h_Q) \langle f \rangle_Q h_Q(P_0) \right|$ is constant on $\hat{P}_0$

$\Rightarrow$ if $A(x) > C_0 \|b\|_{BMO} \langle |f| \rangle_{Q_0}$, then $A(y) > C_0 \|b\|_{BMO} \langle |f| \rangle_{Q_0}$ for all $y \in \hat{P}_0$

$\Rightarrow \overset{\triangleright}{\Pi}_b f(y) \geq A(y) > C_0 \|b\|_{BMO} \langle |f| \rangle_{Q_0}, \forall y \in \hat{P}_0 \Rightarrow \boxed{\hat{P}_0 \subset F}$ **Contradicts maximality of P_0 in F!**

$\Rightarrow$ $A(x) \leq C_0 \|b\|_{BMO} \langle |f| \rangle_{Q_0}$.

$|(b, h_Q)| \lesssim \sqrt{|Q|} \|b\|_{BMO}$

$\to$ $B(x) \leq |(b, h_{\hat{P}_0})| \langle |f| \rangle_{\hat{P}_0} \frac{1}{\sqrt{|\hat{P}_0|}} \lesssim \sqrt{|\hat{P}_0|} \|b\|_{BMO} \langle |f| \rangle_{\hat{P}_0} \frac{1}{\sqrt{|\hat{P}_0|}} = \|b\|_{BMO} \langle |f| \rangle_{\hat{P}_0}$

If $\langle |f| \rangle_{\hat{P}_0} > C_0 \langle |f| \rangle_{Q_0}$, then $M^{\mathcal{D}}_{Q_0} f(x) > C_0 \langle |f| \rangle_{Q_0} \; \forall x \in \hat{P}_0 \Rightarrow \boxed{\hat{P}_0 \subset F}$ **Contradicts maximality of P_0 in F!**

$\Rightarrow \langle |f| \rangle_{\hat{P}_0} \leq C_0 \langle |f| \rangle_{Q_0} \Rightarrow$ $B(x) \lesssim C_0 \|b\|_{BMO} \langle |f| \rangle_{Q_0}$

$$\Rightarrow \begin{cases} x \notin F : |\Pi_{b,Q_0} f(x)| \leq 2 C_0 \|b\|_{BMO} \langle |f| \rangle_{Q_0} \\ x \in F : |\Pi_{b,Q_0} f(x)| \lesssim 2 C_0 \|b\|_{BMO} \langle |f| \rangle_{Q_0} + |\Pi_{b,P_0} f(x)| \\ \qquad P_0 = \text{unique element of } \mathcal{F} \text{ s.t. } x \in P_0 \end{cases}$$

$$\Rightarrow |\Pi_{b,Q_0} f(x)| \lesssim 2 C_0 \|b\|_{BMO} \langle |f| \rangle_{Q_0} + \sum_{R \in \mathcal{F}} |\Pi_{b,R} f(x)|.$$

■

4. Proving $(*)$:

→ The following result (Cristina Pereyra):

PROP.: If $T: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded linear or sublinear operator that satisfies:

$$\operatorname{supp}(Th_Q) \subseteq Q, \ \forall Q \in \mathcal{D}$$

then T is of weak $(1,1)$ type, with:

$$\boxed{|\{x : |Tf(x)| > \alpha\}| \leq C_n \underbrace{\|T\|_{L^2 \to L^2}}_{=:B} \frac{\|f\|_{L^1}}{\alpha}}$$

where C_n is a dimensional constant.

Proof: Let $f \in L^1(\mathbb{R})$ and let $f = g + r$ be the CZ decomposition at level $\frac{\alpha}{B}$:

- $\|g\|_1 \leq \|f\|_1$
- $\|g\|_\infty \leq 2^n \frac{\alpha}{B}$
- $r = \sum_j r_j$, where each r_j is supported in a $Q_j \in \mathcal{D}$ and the cubes Q_j are pairwise disjoint.
- $\int_{Q_j} r = 0$
- $\|r_j\|_1 \leq 2^{n+1} \frac{\alpha}{B} |Q_j|$
- $\sum_j |Q_j| \leq \frac{B}{\alpha} \|f\|_1$.

Then:

$$|\{x : |Tf(x)| > \alpha\}| \leq \underbrace{|\{x : |Tg(x)| > \tfrac{\alpha}{2}\}|}_{① \lesssim \frac{B}{\alpha}\|f\|_1} + \underbrace{|\{x : |Tr(x)| > \tfrac{\alpha}{2}\}|}_{② \lesssim \frac{B}{\alpha}\|f\|_1} \lesssim \frac{B}{\alpha}\|f\|_1.$$

① Since T is of strong type $(2,2)$, it is of weak type $(2,2)$:

by Chebyshev: $|\{x : |Tg(x)| > \frac{\alpha}{2}\}| \leq \frac{1}{(\alpha/2)^2} \int_{|Tg| \geq \alpha/2} |Tg|^2 dx \leq \frac{4}{\alpha^2} \|Tg\|_{L^2}^2$
$\leq \frac{4}{\alpha^2} B^2 \|g\|_{L^2}^2$

But by Hölder: $\|g\|_2 \leq \|g\|_1^{1/2} \|g\|_\infty^{1/2} \leq \left(\|f\|_1 \cdot 2^n \frac{\alpha}{B}\right)^{1/2}$

$$\Rightarrow ① \leq \frac{4}{\alpha^2} B^2 \|f\|_1 2^n \frac{\alpha}{B} \simeq \frac{B}{\alpha} \|f\|_1.$$

② $(r, h_Q) = \sum_j \underbrace{(r_j, h_Q)}_{0 \text{ unless } Q \text{ intersects } Q_j \supseteq \operatorname{supp}(r_j)}$

$Q \subset Q_j$: just (r_j, h_Q)

$Q \supsetneq Q_j$: $h_Q(Q_j) \int_{Q_j} r_j = \underline{\underline{0}}$

$$= \sum_j \left(\sum_{Q \subset Q_j} (r_j, h_Q) \right)$$

$$\Rightarrow Tr(x) = \sum_Q (r, h_Q)(Th_Q)(x) = \sum_j \sum_{Q \subset Q_j} (r, h_Q) \underbrace{(Th_Q)(x)}_{0 \text{ unless } x \in Q}$$

$$\Rightarrow \operatorname{supp}(Tr) \subseteq \cup_j Q_j \Rightarrow |\{x : |Tr(x)| > \alpha/2\}| \leq \sum_j |Q_j| \leq \frac{B}{\alpha} \|f\|_1. \blacksquare$$

Prop.: The operator $\overset{\triangleright}{\Pi}_b$ satisfies:

(1). $\overset{\triangleright}{\Pi}_b$ dominates Π_b: $|\Pi_b f(x)| \le \overset{\triangleright}{\Pi}_b f(x), \ \forall x \in \mathbb{R}^n$ (already shown)

(2). $\overset{\triangleright}{\Pi}_b$ is dominated by $M\Pi_b$: $\boxed{\overset{\triangleright}{\Pi}_b f(x) \le M(\Pi_b f)(x), \ \forall x \in \mathbb{R}^n}$

(3). $\overset{\triangleright}{\Pi}_b$ is strong (2,2): $\boxed{\|\overset{\triangleright}{\Pi}_b f\|_{L^2} \lesssim \|b\|_{BMO_D} \|f\|_{L^2}}$

(4). $\overset{\triangleright}{\Pi}_b$ is weak (1,1): $\boxed{|\{x \in \mathbb{R}^n : \overset{\triangleright}{\Pi}_b f(x) > \alpha\}| \le \frac{C_n}{\alpha} \|b\|_{BMO_D} \|f\|_1.}$

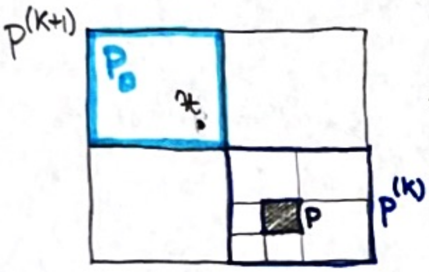
Proof:

(2). Let $P \in D$ and define $F_P(x) := \sum_{Q \supsetneq P} (b, h_Q) \langle f \rangle_Q h_Q(x)$.

Two possibilities for x:

- $x \in P \Rightarrow |F_P(x)| = \left| \sum_{Q \supsetneq P} (b, h_Q) \langle f \rangle_Q h_Q(P) \right| \cdot \mathbb{1}_P(x) = |\langle \Pi_b f \rangle_P| \mathbb{1}_P(x) \le \langle |\Pi_b f| \rangle_P \mathbb{1}_P(x)$
$\le M_D(\Pi_b f)(x)$
- $x \notin P \Rightarrow$ there exists a unique integer $K \ge 0$ s.t.

$x \in P^{(K+1)} \setminus P^{(K)} \Rightarrow$ there is a unique $P_0 \in (P^{(K+1)})_{(1)}$, $P_0 \ne P^{(K)}$ such that: $x \in P_0$



$$\Rightarrow F_P(x) = (b, h_{P^{(K+1)}}) \langle f \rangle_{P^{(K+1)}} h_{P^{(K+1)}}(x) + \sum_{Q \supsetneq P^{(K+1)}} (b, h_Q) \langle f \rangle_Q h_Q(P^{(K+1)})$$

$$= (b, h_{P^{(K+1)}}) \langle f \rangle_{P^{(K+1)}} h_{P^{(K+1)}}(P_0) + \sum_{Q \supsetneq P^{(K+1)}} (b, h_Q) \langle f \rangle_Q h_Q(P_0)$$

$$= \left(\sum_{Q \supsetneq P_0} (b, h_Q) \langle f \rangle_Q h_Q(P_0) \right) \mathbb{1}_{P_0}(x)$$

$$= \langle \Pi_b f \rangle_{P_0} \mathbb{1}_{P_0}(x) \Rightarrow |F_P(x)| \le \langle |\Pi_b f| \rangle_{P_0} \mathbb{1}_{P_0}(x)$$
$$\le M_D(\Pi_b f)(x)$$

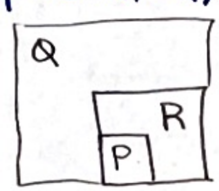
$\Rightarrow |F_P(x)| \le M(\Pi_b f)(x), \ \forall x \in \mathbb{R}, P \in D$

$\Rightarrow \overset{\triangleright}{\Pi}_b f(x) \le M(\Pi_b f)(x)$ //

(3). Follows from (2): $\|\overset{\triangleright}{\Pi}_b f\|_{L^2} \le \|M(\Pi_b f)\|_{L^2} \lesssim \|\Pi_b f\|_{L^2} \lesssim \|b\|_{BMO} \|f\|_{L^2}$ //

$\boxed{\|\Pi_b f\|_{L^2} \lesssim \|b\|_{BMO} \|f\|_{L^2}}$ $\to$ will show later. In fact $\|\Pi_b : L^2 \to L^2\| \simeq \|b\|_{BMO}$

(4). Follows from the previous result as soon as we show that $\operatorname{supp}(\overset{\triangleright}{\Pi}_b h_Q) \subset Q, \ \forall Q \in D$:



$$\overset{\triangleright}{\Pi}_b h_Q(x) = \sup_{P \in D} \underbrace{\left| \sum_{R \supsetneq P} (b, h_R) \underbrace{\langle h_Q \rangle_R}_{0 \text{ unless } Q \supsetneq R, \text{ then it's } h_Q(R)} h_R(x) \right|}_{0 \text{ for all } P \supseteq Q}$$

$$= \sup_{P \subsetneq Q} \underbrace{\left| \sum_{\substack{R \supsetneq P \\ R \subsetneq Q}} (b, h_R) h_Q(R) h_R(x) \right|}_{0 \text{ if } x \notin Q} \Rightarrow \operatorname{supp}(\overset{\triangleright}{\Pi}_b h_Q) \subset Q. \ //$$

■